# 4. PERMUTATIONS AND COMBINATIONS

## Quick Review

1. An arrangement that can be formed by taking some or all of a finite set of things (or objects) is called a ***permutation***.
2. A permutation is said to be a ***linear permutation*** if the objects are arranged in a line. A linear permutation is simply called as a permutation.
3. A permutation is said to be a ***circular permutation*** if the objects are arranged in the form of a circle (a closed curve).
4. The number of (linear) permutations that can be formed by taking r things at a time from a set of n dissimilar things ($r \le n$) is denoted by ${}^nP_r$ or P(n, r) or $P\begin{pmatrix} n \\ r \end{pmatrix}$.
5. The number of permutations of n dissimilar things taken r at a time is equal to the number of ways of filling of r blank places arranged in a row by n dissimilar things.
6. **Fundamental counting principle :** If an operation can be performed in m ways and a second operation can be performed in n ways corresponding to each performance of the first operation, then the two operations in succession can be performed in mn ways.
7. If in k operations, the first operation can be performed in $n_1$ ways, the second operation can be performed in $n_2$ ways, third operation can be performed in $n_3$ ways and so on, then the k operations in succession can be performed in $n_1\, n_2\, n_3 \ldots n_k$ ways.
8. ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$.
9. If n is a non-negative integer, then ***factorial*** n is denoted by n! or $\angle n$ and defined as follows.

(i) $0! = 1$; (ii) If $n > 0$ then $n! = n \times (n-1)!$

10. If n is a positive integer, then n! is the product of first n positive integers. i.e., $n! = 1.2.3\ldots n$.
11. ${}^nP_r = \dfrac{n!}{(n-r)!}$
12. The number of permutations of n dissimilar things taken all at a time is ${}^nP_n = n!$.
13. ${}^nP_r = {}^{(n-1)}P_r + r\,{}^{(n-1)}P_{r-1}$.
14. The number of injections (one-one functions) that can be defined from a set containing r elements into a set containing n elements is ${}^nP_r$.
15. The number of bijections (one-one onto functions) that can be defined from a set containing n elements onto a set containing n elements is n!.
16. The number of permutations of n dissimilar things taken r at a time when repetition of things is allowed any number of times is $n^r$.
17. The number of permutations of n dissimilar things taken not more than r at a time, when each thing may occur any number of times is $\dfrac{n(n^r-1)}{n-1}$.
18. The number of functions that can defined from a set containing r elements into a set containing n elements is $n^r$.

19. The number of permutations of n things taken all at a time when p of them are all alike and the rest all different is $\frac{n!}{p!}$.

20. If $p_1$ things are alike of one kind, $p_2$ things are alike of second kind, $p_3$ things are alike of third kind and so on, $p_k$ things are alike of $k^{th}$ kind in $p_1+p_2+\ldots+p_k$ things, then the number of permutations obtained by taking all the things is $\frac{(p_1+p_2+\ldots+p_k)!}{p_1!p_2!\ldots p_k!}$.

21. The number of circular permutations on n different things taken r at a time is $\frac{^nP_r}{r}$.

22. The number of circular permutations of n different things taken all at a time is (n – 1)!.

23. The number of circular permutations of n things taken r at a time in one direction is $\frac{^nP_r}{2r}$.

24. The number of circular permutations of n things taken all at a time in one direction is $\frac{1}{2}(n-1)!$.

25. (i) $^nP_r + r.^nP_{r-1} = {}^{(n+1)}P_r$

(ii) $\frac{^nP_r}{^{n-1}P_{r-1}} = n$

(iii) $\frac{^nP_r}{^nP_{r-1}} = n-r+1$.

26. If n is a positive integer and p is a prime number then the exponent of p in n! is $\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots$ where [x] denotes the greatest integer ≤ x.

27. The number of ways in which m (first type of different) things and n(second type of different) things (m + 1 ≥ n) can be arranged in a row so that no two things of second kind come together is m! $^{(m+1)}P_n$.

28. The number of ways in which m (first type of different) things and n(second type of different) things can be arranged in a row so that all the second type of things come together is n! (m+1)!.

29. The number of ways in which n(first type of different) things and n(second type of different) things can be arranged in a row alternatively is 2 × n! × n!.

30. Sum of the numbers formed by taking all the given n digits (excluding 0) is (Sum of all the n digits) × (n – 1)! × (111…n times).

31. Sum of the numbers formed by taking all the given n digits (including 0) is (Sum of all the n digits)
[(n – 1)! × (111…n times) – (n – 2)! (111…(n – 1) times)].

32. Sum of all r-digit numbers formed by taking the given n digits (without zero) is (sum of all the n digits) × $^{n-1}P_{r-1}$ × (111…r times).

33. Sum of all the r-digit numbers formed by taking the given n digits (including 0) is (sum of all the n digits) [$^{n-1}P_{r-1}$ × (111………r times) – $^{n-2}P_{r-2}$ × {111…(r–1) times}].

34. The number of ways in which m (first type of different) things and n (second type of different) things, (m ≥ n) can be arranged in a circle so that no two things of second kind come together is $(m-1)!\ {}^{m}P_{n}$.
35. The number of ways in which m (first type of different) things and n(second type of different) things can be arranged in a circle so that all the second type of things come together is m! n!.
36. The number of ways in which m (first type of different) things and n(second type of different) things can be arranged in the form of garland so that all the second type of things come together is m! n!/2.
37. A selection that can be formed by taking some or all of a finite set of things (or objects) is called a ***combination***.
38. Formation of a combination by taking r elements from a finite set A means picking up an r element subset of A.
39. The number of combinations of n dissimilar things taken r at a time is equal to the number of r element subsets of a set containing n elements.
40. The number of combinations of n dissimilar things taken r at a time is denoted by ${}^{n}C_{r}$ or C(n, r) or $C\binom{n}{r}$ or $\binom{n}{r}$.
41. ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
42. ${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n(n-1)(n-2)...(n-r+1)}{1.2.3....r}$.
43. ${}^{n}C_{r} = {}^{n}C_{n-r}$.
44. ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{(n+1)}C_{r}$.
45. If ${}^{n}C_{r} = {}^{n}C_{s}$, then r = s or r + s = n.
46. The number of diagonals in a regular polygon of n sides is ${}^{n}C_{2} - n = \frac{n(n-3)}{2}$.
47. The number of ways in which (m + n) things can be divided into two different groups of m and n things respectively is $\frac{(m+n)!}{m!n!}$.
48. The number of ways in which 2n things can be divided into two equal groups of n things each is $\frac{(2n)!}{2!(n!)^2}$.
49. The number of ways in which $(n_1 + n_2 + \ldots n_k)$ things can be divided into k different groups of $n_1$ things, $n_2$ things, $n_3$ things, ………… $n_k$ things respectively is $\frac{(n_1 + n_2 + ... + n_k)!}{n_1!n_2!...n_k!}$.
50. The number of ways in which kn things can be divided into k equal groups of n things each is $\frac{(kn)!}{k!(n!)^k}$.
51. The total number of combinations of (p + q) things taken any number at a time when p things are alike of one kind and q things are alike of a second kind is (p + 1) (q + 1).

52. The total number of combinations of p + q things taken any number at a time, includes the case in which nothing will be selected.

53. The total number of combinations of (p + q) things taken one or more at a time when p things are alike of one kind and q things are alike of a second kind is (p + 1)(q + 1) – 1.

54. The total number of combinations of $(p_1+p_2+\ldots+p_k)$ things taken any number at a time when $p_1$ things are alike of one kind, $p_2$ things are alike of a second kind, … $p_k$ things are alike of kth kind, is $(p_1 + 1)(p_2 + 1) \ldots (p_k + 1)$.

55. The total number of combinations of $(p_1+p_2+\ldots+p_k)$ things taken one or more at a time when $p_1$ things are alike of one kind, $p_2$ things are alike of a second kind, … $p_k$ things are alike of kth kind, is $(p_1 + 1)(p_2 + 1) \ldots (p_k + 1) - 1$.

56. The total number of combinations of n different things taken any number at a time is $2^n$.

57. The total number of combinations of n different things taken one or more at a time is $2^n - 1$.

58. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots {}^nC_n = 2^n$.

59. If n is a positive integer, then n can be uniquely expressed as n = $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ where $p_1, p_2, \ldots p_k$ are primes in increasing order and $\alpha_1, \alpha_2, \ldots \alpha_k$ are non-negative integers. This representation of n is called ***prime factorisation*** of n in ***canonical form*** or ***prime power factorisation*** of n.

60. The number of positive divisors of a positive integer n = $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ (the prime factorisation) is $(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) \ldots (\alpha_k + 1)$.

61. (i) ${}^nC_r = {}^nC_{n-r}$ (ii) ${}^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$ (iii) ${}^nC_r = {}^nC_s \Rightarrow$ r = s or r + s = n

62. (i) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$ (ii) $\dfrac{{}^nC_r}{{}^{(n-1)}C_r} = \dfrac{n}{n-r}$ (iii) $\dfrac{{}^nC_r}{{}^{(n-1)}C_{r-1}} = \dfrac{n}{r}$

63. The number of parallelograms formed when a set of m parallel lines are intersecting another set of n parallel lines is ${}^mC_2 \times {}^nC_2$.

64. If there are n points in a plane no three of which are on the same straight line excepting p points which are collinear, then

(i) the number of straight lines formed by joining them is ${}^nC_2 - {}^pC_2 + 1$.

(ii) the number of triangles formed by joining them is ${}^nC_3 - {}^pC_3$.

65. The number of ways that n sovereigns can be given away when there are k applicants and any applicant may have either 0, 1, 2, 3, 4, 5 …… or n sovereigns is ${}^{(n+k-1)}C_{k-1}$.

66. (i) The number of ways in which exactly r letters can be placed in wrongly addressed envelopes when n letters are putting in n addressed envelopes is ${}^nP_r\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^r \frac{1}{r!}\right]$.

(ii) The number of ways in which n different letters can be put in their n addressed envelopes so that all the letters are in the wrong envelopes = $n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right]$.

67. (i) The number of ways of answering one or more of n questions is $2^n - 1$.

(ii) The number of ways of answering one or more of n questions when each question have an alternative is $3^n - 1$.

(iii) The number of ways of answering all of n questions when each question have an alternative is $2^n$.

68. The number of distinct positive integral divisors of $p_1^{k_1}p_2^{k_2}...p_r^{k_r}$ where $p_1$, $p_2$, …,$p_r$ are primes in ascending order, is $(k_1 + 1)(k_2 + 1) \ldots (k_r + 1)$.

69. The sum of distinct positive integral divisors of $p_1^{k_1}p_2^{k_2}...p_r^{k_r}$ where $p_1$, $p_2$, …,$p_r$ are primes in ascending order, is $\frac{p_1^{k_1+1}-1}{p_1-1}\cdot\frac{p_2^{k_2+1}-1}{p_2-1}....\frac{p_r^{k_r+1}-1}{p_r-1}$.